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Matchings in simplicial complexes, circuits and toric varieties

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Abstract

Using a generalized notion of matching in a simplicial complex and circuits of vector configurations, we compute lower bounds for the minimum number of generators, the binomial arithmetical rank and the A -homogeneous arithmetical rank of a lattice ideal. Prime lattice ideals are toric ideals, i.e. the defining ideals of toric varieties.

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1. Introduction

In this article we introduce a generalized notion of matching in a simplicial complex, called J -matching, and use it to define certain combinatorial invariants of the complex. In several cases they correspond to well-known numbers in graph theory, like the matching number, the chromatic number and the cardinality of a minimum edge covering of a graph. The invariants defined here arise in connection with a problem coming from toric varieties, namely the computation of the arithmetical rank of a lattice ideal.

Recently Katsabekis, Morales and Thoma in [9] associated to every lattice ideal $I_{L,\rho}$ a cone σ and a simplicial complex \mathcal{D}_σ with vertices the minimal nonfaces of σ . As they have shown, combinatorial invariants of \mathcal{D}_σ provide lower bounds for the minimum number of generators,

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the binomial arithmetical rank and the A -homogeneous arithmetical rank of a lattice ideal. Even more these numbers provide information about the complexity of the problem of computing the arithmetical rank of a lattice ideal.

In Section 2 we introduce J -matchings as well as the invariants $\delta(\mathcal{D})_J$ and relate them with known combinatorial numbers.

In Section 3 we present the basic theory of lattice ideals. Also we make the connection between the invariants $\delta(\mathcal{D})_J$ and the radical of a lattice ideal.

In Section 4 we study circuits of a vector configuration A and the related simplicial complex Δ_A . Circuits can be computed using linear algebra and they provide all the necessary information to construct Δ_A . It is proved that the simplicial complexes Δ_A and \mathcal{D}_σ are identical when A is an extremal vector configuration. Using this fact we compute some of the invariants $\delta(\mathcal{D})_J$ which provide lower bounds for the minimum number of generators, the binomial arithmetical rank and the A -homogeneous arithmetical rank of a lattice ideal. Finally we present an example which explains how the techniques of this article can be applied to compute the exact values of the binomial arithmetical rank and the A -homogeneous arithmetical rank of certain lattice ideals.

2. Matchings in simplicial complexes

Let $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ be a finite set. An *abstract simplicial complex* \mathcal{D} on the vertex set \mathcal{V} is a collection of subsets of \mathcal{V} satisfying:

- (i) every element of \mathcal{V} is in \mathcal{D} ,
- (ii) if $T \in \mathcal{D}$ and $G \subset T$, then $G \in \mathcal{D}$.

A set $T \subset \mathcal{D}$ of cardinality $m + 1$ has *dimension* $m \geq -1$ and is called an m -*simplex* of \mathcal{D} . The empty set \emptyset is the unique simplex of dimension -1 in any simplicial complex \mathcal{D} . The 0-simplices of \mathcal{D} are called *vertices*, while the 1-simplices are called *edges*. The *dimension* $\dim(\mathcal{D})$ of \mathcal{D} is the maximum of the dimensions of its simplices. Let Ω be the set $\{0, 1, \dots, \dim(\mathcal{D})\}$ and $J \subset \Omega$.

Definition 2.1. A set $\mathcal{M} = \{T_1, \dots, T_s\}$ of simplices of \mathcal{D} is called a J -*matching* in \mathcal{D} if $T_k \cap T_l = \emptyset$ for every $1 \leq k, l \leq s$ and $\dim(T_k) \in J$ for every $1 \leq k \leq s$. Let $\text{supp}(\mathcal{M}) = \bigcup_{i=1}^s T_i$, which is a subset of the vertices \mathcal{V} . A J -matching \mathcal{M} in \mathcal{D} is called a *perfect matching* if $\text{supp}(\mathcal{M}) = \mathcal{V}$. We denote by $\text{card}(\mathcal{M})$ the cardinality s of the set \mathcal{M} .

Definition 2.2. A J -matching \mathcal{M} in \mathcal{D} is called a *maximal* J -matching if $\text{supp}(\mathcal{M})$ has the maximum possible cardinality among all J -matchings. By $\delta(\mathcal{D})_J$ we denote the minimum $\text{card}(\mathcal{M})$ among all maximal J -matchings \mathcal{M} in \mathcal{D} .

Remark 2.3. Note that for $J = \{0\}$ we have $\delta(\mathcal{D})_{\{0\}} = n$, where n is the number of vertices of \mathcal{D} .

Also it follows, from the definitions, that if $\mathcal{D} = \bigcup_{i=1}^t \mathcal{D}_i$ then

$$\delta(\mathcal{D})_J = \sum_{i=1}^t \delta(\mathcal{D}_i)_J,$$

where \mathcal{D}_i are the connected components of \mathcal{D} . This means that δ is additive on connected components.

Example 2.4. Let \mathcal{D} be the simplicial complex on $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ consisting of all subsets of the sets $\{v_1, v_2, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_6\}$. We have that $\delta(\mathcal{D})_{\{0,1,2\}} = \delta(\mathcal{D})_{\{1\}} = 3$ which are attained by the maximal $\{0, 1, 2\}$ ($\{1\}$)-matching $\{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}\}$. Note also that $\delta(\mathcal{D})_{\{0,2\}} = 4$ which is attained by the maximal $\{0, 2\}$ -matching $\{\{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}, \{v_6\}\}$. Also $\delta(\mathcal{D})_{\{0\}} = 6$ and $\delta(\mathcal{D})_{\{2\}} = 1$, which is attained by the maximal $\{2\}$ -matching $\{\{v_1, v_2, v_3\}\}$. Note that the last matching is not perfect.

Remark 2.5. If $0 \in J$, then every maximal J -matching in \mathcal{D} is perfect.

Since the support of every maximal $\{q\}$ -matching \mathcal{M} in \mathcal{D} has the same cardinality and consists only of q -simplices, we have that $\text{card}(\mathcal{M})$ is equal to $\delta(\mathcal{D})_{\{q\}}$ for every maximal $\{q\}$ -matching \mathcal{M} .

A *simple graph* G is an abstract simplicial complex consisting only of vertices and edges, i.e. $\dim(G) \leq 1$. The *complement* of G , denoted by \bar{G} , is the graph with the same vertices as G , and there is an edge between the vertices v_i and v_j if and only if there is no edge between v_i and v_j in the graph G . An *isolated* vertex of G is a vertex not joined with any other vertex by an edge. A subset T of the vertices of a simple graph G is called *independent* if every pair of elements in T are not joined by an edge. On the other hand, T is called a *clique* if every pair of elements in T are joined by an edge. Given a simple graph G , with vertices $\{v_1, \dots, v_n\}$, and an integer k , a k -coloring of G is a function $c: V \rightarrow \{1, \dots, k\}$ such that $c(v_i) \neq c(v_j)$ if the vertices v_i, v_j are joined by an edge. The *chromatic number* $\gamma(G)$ of G is the smallest integer k such that there is a k -coloring of G .

A subset M of the edges of a simple graph G is called a *matching* in G if there are no two edges which are incident with a common vertex. M is a *maximum matching* if it has the maximum possible cardinality among all matchings. The cardinality of a maximum matching in G is commonly known as its *matching number* and will be denoted by $\tau(G)$.

An *edge covering* of G is a subset M of the edges of G such that each vertex of G is an end of some edge in M . Note that edge coverings do not always exist; a simple graph G has an edge covering if and only if it has no isolated vertices. A *minimum edge covering* of G is an edge covering with the minimum possible cardinality, which is denoted by $\beta(G)$.

To every simplicial complex \mathcal{D} we can associate a simple graph, called the *1-skeleton* of \mathcal{D} and denoted by $\mathbb{G}(\mathcal{D})$, formed by the simplices of \mathcal{D} of dimension at most 1. To every simple graph G we can associate the *clique complex* $\mathbb{D}(G)$ of G , which is a simplicial complex with q -simplices the cliques with $q + 1$ vertices of the graph G . Note that \mathcal{D} is a subcomplex of $\mathbb{D}(\mathbb{G}(\mathcal{D}))$ and $\mathbb{G}(\mathbb{D}(G)) = G$.

Proposition 2.6. The matching number $\tau(\mathbb{G}(\mathcal{D}))$ of the graph $\mathbb{G}(\mathcal{D})$ is equal to $\delta(\mathcal{D})_{\{1\}}$.

Proof. As we noticed in Remark 2.5, every maximal $\{1\}$ -matching in $\mathbb{G}(\mathcal{D})$ consists of $\delta(\mathcal{D})_{\{1\}}$ edges, since we use only 1-simplices of \mathcal{D} . \square

The proof of the next proposition is a special case of the proof of Proposition 3.3 and therefore it is postponed to Section 3.

Proposition 2.7. In the case that $\mathbb{G}(\mathcal{D})$ has no isolated vertices it holds

$$\delta(\mathcal{D})_{\{0,1\}} = \beta(\mathbb{G}(\mathcal{D})).$$

The next identity is a generalization of Gallai's identity (see [8]).

Theorem 2.8. *Let $q \geq 0$ be an integer. The following equality is true:*

$$q\delta(\mathcal{D})_{\{q\}} + \delta(\mathcal{D})_{\{0,q\}} = \delta(\mathcal{D})_{\{0\}}.$$

Proof. Let $\{T_1, \dots, T_s, \{v_1\}, \dots, \{v_t\}\}$ be a maximal $\{0, q\}$ -matching in \mathcal{D} such that $\delta(\mathcal{D})_{\{0,q\}} = s + t$ and $\dim(T_i) = q$. Since a maximal $\{0, q\}$ -matching is perfect we conclude that $s(q + 1) + t = \delta(\mathcal{D})_{\{0\}}$. We claim that $\{T_1, \dots, T_s\}$ is a maximal $\{q\}$ -matching in \mathcal{D} . Certainly $\{T_1, \dots, T_s\}$ is a $\{q\}$ -matching in \mathcal{D} . Suppose that $\{T_1, \dots, T_s\}$ is not maximal, then there exists a maximal $\{q\}$ -matching $\{B_1, \dots, B_r\}$ in \mathcal{D} with $r > s$. Extend this matching to a maximal $\{0, q\}$ -matching in \mathcal{D} by adding the remaining p vertices of \mathcal{D} not in $\text{supp}(\bigcup_{i=1}^r B_i)$. In this way we take a perfect $\{0, q\}$ -matching in \mathcal{D} , so $r(q + 1) + p = \delta(\mathcal{D})_{\{0\}}$. Now the relation $sq + s + t = rq + r + p$ together with the fact that $r > s$ give

$$\delta(\mathcal{D})_{\{0,q\}} = s + t = r + p + (r - s)q > r + p,$$

which is impossible since $\delta(\mathcal{D})_{\{0,q\}}$ is the minimum $\text{card}(\mathcal{M})$ among all maximal $\{0, q\}$ -matchings \mathcal{M} in \mathcal{D} . Thus $\{T_1, \dots, T_s\}$ is a maximal $\{q\}$ -matching in \mathcal{D} , so, from Remark 2.5, the above matching consists of $\delta(\mathcal{D})_{\{q\}}$ simplices. Therefore $s = \delta(\mathcal{D})_{\{q\}}$. Finally, from the relation $sq + s + t = \delta(\mathcal{D})_{\{0\}}$ we get $q\delta(\mathcal{D})_{\{q\}} + \delta(\mathcal{D})_{\{0,q\}} = \delta(\mathcal{D})_{\{0\}}$. \square

Remark 2.9. Applying Propositions 2.6, 2.7 and Theorem 2.8 in the special case that $\mathbb{G}(\mathcal{D})$ has no isolated vertices we take the Gallai's identity $\tau(\mathbb{G}(\mathcal{D})) + \beta(\mathbb{G}(\mathcal{D})) = n$.

Proposition 2.10. *Let $\mathbb{D}(G)$ be the clique complex of a simple graph G . Then $\delta(\mathbb{D}(G))_{\Omega}$ equals $\gamma(\bar{G})$, the chromatic number of the complement of G .*

Proof. Suppose that $\delta(\mathbb{D}(G))_{\Omega} = s$ and $\gamma(\bar{G}) = k$. There exists a maximal Ω -matching $\{T_1, \dots, T_s\}$ in $\mathbb{D}(G)$, which covers all the vertices since $0 \in \Omega$. Every T_i is a clique of the graph G , so the vertices of T_i consist an independent subset of \bar{G} . Thus they can be colored by the same color in \bar{G} . This means that the graph \bar{G} is s -colorable. So $k \leq s$. Now consider a k -coloring of \bar{G} , then the k colors induce a partition $\{V_1, \dots, V_k\}$ of the vertices of \bar{G} such that the vertices in V_i have the same color. Thus each V_i is independent in \bar{G} , which implies that every V_i is a clique in G and therefore $V_i \in \mathbb{D}(G)$. Also $\{V_1, \dots, V_k\}$ is a maximal Ω -matching in $\mathbb{D}(G)$, since $V_i \cap V_j = \emptyset$ for every $1 \leq i, j \leq k$, $\dim(V_i) \in \Omega$ and it covers all the vertices of $\mathbb{D}(G)$. Consequently $s \leq k$ and therefore $s = k$. \square

Lemma 2.11. *Let $0 \in J$ and $\mathcal{D}' \subset \mathcal{D}$ be two simplicial complexes on the same vertex set, then $\delta(\mathcal{D}')_J \geq \delta(\mathcal{D})_J$.*

Proof. Suppose that $\{T_1, \dots, T_s\}$ is a maximal J -matching in \mathcal{D}' with $s = \delta(\mathcal{D}')_J$. Note that since $0 \in J$ any maximal J -matching is perfect. Thus $\{T_1, \dots, T_s\}$ is also a maximal J -matching in \mathcal{D} and therefore $s \geq \delta(\mathcal{D})_J$. \square

Corollary 2.12. *For a simplicial complex \mathcal{D} we have $\delta(\mathcal{D})_{\Omega} \geq \gamma(\overline{\mathbb{G}(\mathcal{D})})$.*

Proof. The simplicial complexes $\mathcal{D} \subset \mathbb{D}(\mathbb{G}(\mathcal{D}))$ have the same vertex set, so, from Lemma 2.11, we take that $\delta(\mathcal{D})_{\Omega} \geq \delta(\mathbb{D}(\mathbb{G}(\mathcal{D})))_{\Omega}$. But $\delta(\mathbb{D}(\mathbb{G}(\mathcal{D})))_{\Omega} = \gamma(\overline{\mathbb{G}(\mathcal{D})})$, from Proposition 2.10, which gives the demanded relation. \square

3. Applications of J -matchings to toric varieties

3.1. Basics on lattice ideals

Let K be an algebraically closed field. A *lattice* is a finitely generated free abelian group. A *partial character* (L, ρ) on \mathbb{Z}^m is a homomorphism ρ from a sublattice L of \mathbb{Z}^m to the multiplicative group $K^* = K - \{0\}$. Given a partial character (L, ρ) on \mathbb{Z}^m , the ideal

$$I_{L,\rho} := (\{\mathbf{x}^{\alpha_+} - \rho(\alpha)\mathbf{x}^{\alpha_-} \mid \alpha = \alpha_+ - \alpha_- \in L\}) \subset K[x_1, \dots, x_m]$$

is called *lattice ideal*. Where $\alpha_+ \in \mathbb{N}^m$ and $\alpha_- \in \mathbb{N}^m$ denote the positive and negative part of α , respectively, and $\mathbf{x}^\beta = x_1^{b_1} \cdots x_m^{b_m}$ for $\beta = (b_1, \dots, b_m) \in \mathbb{N}^m$. Lattice ideals are binomial ideals, since they are generated by binomials.

We assume that L is a nonzero *positive* sublattice of \mathbb{Z}^m , that is $L \cap \mathbb{N}^m = \{\mathbf{0}\}$. This means that the lattice ideal $I_{L,\rho}$ is homogeneous with respect to some positive grading.

If $L = \langle \mathbf{l}_1, \dots, \mathbf{l}_k \rangle$ is a sublattice of \mathbb{Z}^m of rank $k < m$, then there exists a matrix $\mathbf{D} \in \mathbb{Z}^{(m-k) \times m}$ of rank $m - k$ such that

$$L \subset \ker_{\mathbb{Z}}(\mathbf{D}) := \{\mathbf{u} \in \mathbb{Z}^m : \mathbf{D}\mathbf{u}^T = \mathbf{0}^T\}.$$

Set $\mathbf{L} = (\mathbf{l}_1, \dots, \mathbf{l}_k)$ the matrix with columns $\mathbf{l}_1, \dots, \mathbf{l}_k$, then there are unimodular integral matrices \mathbf{U} and \mathbf{Q} of orders m and k , respectively, such that $\mathbf{ULQ} = \text{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$ is in Smith normal form. Where $\lambda_1, \dots, \lambda_k$ are natural numbers and λ_i divides λ_{i+1} . The matrix \mathbf{D} can be chosen as the last $m - k$ rows of \mathbf{U} .

When $L = \ker_{\mathbb{Z}}(\mathbf{D})$, the ideal I_L is prime and called *toric ideal*. In this case the set of zeros $V(I_L)$ in K^m is an *affine toric variety* in the sense of [11], since we do not require normality.

Let $A = \{\mathbf{a}_i \mid 1 \leq i \leq m\}$ be the set of columns of \mathbf{D} , we associate to the lattice ideal $I_{L,\rho}$ the rational polyhedral cone $\sigma = \text{pos}_{\mathbb{Q}}(A) := \{d_1\mathbf{a}_1 + \cdots + d_m\mathbf{a}_m \mid d_i \in \mathbb{Q} \text{ and } d_i \geq 0\}$. The *dimension* of σ is the dimension of the \mathbb{Q} -vector space

$$\text{span}_{\mathbb{Q}}(\sigma) = \{d_1\mathbf{a}_1 + \cdots + d_m\mathbf{a}_m \mid d_i \in \mathbb{Q}\}.$$

For a subset E of $\{1, \dots, m\}$ we denote by σ_E the subcone $\text{pos}_{\mathbb{Q}}(\mathbf{a}_i \mid i \in E)$ of σ . We adopt the convention that $\sigma_{\emptyset} = \{\mathbf{0}\}$. The *relative interior* of σ_E , denoted by $\text{relint}_{\mathbb{Q}}(\sigma_E)$, is the set of all positive rational linear combinations of \mathbf{a}_i , $i \in E$. A *face* \mathcal{F} of σ is any set of the form

$$\mathcal{F} = \sigma \cap \{\mathbf{x} \in \mathbb{Q}^{m-k} : \mathbf{c}\mathbf{x} = 0\},$$

where $\mathbf{c} \in \mathbb{Q}^{m-k}$ and $\mathbf{c}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \sigma$. Faces of dimension one are called *extreme rays*. A cone σ is *strongly convex* if $\sigma \cap -\sigma = \{\mathbf{0}\}$. The condition that the lattice L is positive, is equivalent with the condition that the cone σ is strongly convex.

We grade $K[x_1, \dots, x_m]$ by setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \dots, m$. We define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A,$$

where $\mathbb{N}A$ is the semigroup generated by A . A polynomial $F \in K[x_1, \dots, x_m]$ is called *A-homogeneous* if the monomials in each nonzero term of F have the same A -degree. An ideal I is *A-homogeneous* if it is generated by A -homogeneous polynomials. The lattice ideal $I_{L,\rho}$ is A -homogeneous as well as all lattice ideals with the same saturation. Note also that binomials and trinomials of $I_{L,\rho}$ are always A -homogeneous. The *arithmetical rank* of an ideal I (written $\text{ara}(I)$) is the smallest integer s for which there exist polynomials f_1, \dots, f_s in I such that

$\text{rad}(I) = \text{rad}(f_1, \dots, f_s)$. The computation of the arithmetical rank of a lattice ideal is a difficult problem and remains open even in very simple cases, see [6, Chapter 15]. A usual approach to this problem is to restrict to a certain class of polynomials and ask how many polynomials from this class can generate the radical of the lattice ideal up to radical. Thus, depending on the restrictions of the polynomials, one arrives in various kinds of arithmetical ranks, which are upper bounds of the usual arithmetical rank. For example, if all the polynomials f_1, \dots, f_s are A -homogeneous, the smallest integer s is called A -homogeneous arithmetical rank of $I_{L,\rho}$ and will be denoted by $\text{ara}_A(I_{L,\rho})$. Furthermore requiring that each one of the A -homogeneous polynomials f_1, \dots, f_s has at most $q \geq 2$ nonzero terms, we arrive at the notion of $\text{ara}_{A,q}(I_{L,\rho})$. The integer $\text{ara}_{A,2}(I_{L,\rho})$ is commonly known as the *binomial arithmetical rank* of $I_{L,\rho}$, denoted by $\text{bar}(I_{L,\rho})$. From the definitions, the generalized Krull's principal ideal theorem and the graded version of Nakayama's Lemma we deduce the following inequalities for a lattice ideal $I_{L,\rho}$:

$$\text{ht}(I_{L,\rho}) \leq \text{ara}(I_{L,\rho}) \leq \text{ara}_A(I_{L,\rho}) \leq \text{ara}_{A,q}(I_{L,\rho}) \leq \mu(I_{L,\rho}),$$

where $q \geq 2$. Also $\text{ara}_{A,q+l}(I_{L,\rho}) \leq \text{ara}_{A,q}(I_{L,\rho})$ for any positive integer l .

3.2. J -matchings and toric varieties

Let $\sigma = \text{pos}_{\mathbb{Q}}(\mathbf{r}_1, \dots, \mathbf{r}_t) \subset \mathbb{Q}^n$ be a strongly convex rational polyhedral cone, where $\{\mathbf{r}_1, \dots, \mathbf{r}_t\}$ is a set of integer vectors, one for each extreme ray of σ . The vectors \mathbf{r}_i are called *extreme vectors* of σ . For the rest of this section we are going to deal only with cones σ_E , which are not faces of the cone σ . They form a poset ordered by inclusion. Let $\{\sigma_{\mathbb{E}_1}, \dots, \sigma_{\mathbb{E}_f}\}$ be the minimal elements of this poset, which are called the *minimal nonfaces* of σ . Translating Theorem 3.2 of [9] we have the following theorem:

Theorem 3.1. *The cone σ_E is a minimal nonface of σ if and only if*

- (i) σ_E is not a face of σ ,
- (ii) $\sigma_{E'}$ is a face of σ for every $E' \subset E$,
- (iii) the set $\{\mathbf{r}_i \mid i \in E\}$ is linearly independent.

Definition 3.2. We associate to the cone σ a simplicial complex \mathcal{D}_{σ} with vertices the set $\{\mathbb{E}_1, \dots, \mathbb{E}_f\}$ where $\{\sigma_{\mathbb{E}_1}, \dots, \sigma_{\mathbb{E}_f}\}$ are the minimal nonfaces of σ . Let T be a subset of $\{\mathbb{E}_1, \dots, \mathbb{E}_f\}$, then $T \in \mathcal{D}_{\sigma}$ if

$$\bigcap_{\sigma_{\mathbb{E}_i} \in T} \text{relint}_{\mathbb{Q}}(\sigma_{\mathbb{E}_i}) \neq \emptyset.$$

For a simplicial complex \mathcal{D} we denote by $c_{\mathcal{D}}$ the smallest number s of simplices T_i of \mathcal{D} , such that the subcomplex $\bigcup_{i=1}^s T_i$ is spanning. By $c_{\mathcal{D},q}$ the smallest number s of simplices T_i of \mathcal{D} , such that the subcomplex $\bigcup_{i=1}^s T_i$ is spanning and $\dim(T_i) \leq q-1$. Note that for $q \geq \dim(\mathcal{D})+1$ we have $c_{\mathcal{D},q} = c_{\mathcal{D}}$.

Proposition 3.3. *For a simplicial complex \mathcal{D} we have that $c_{\mathcal{D}} = \delta(\mathcal{D})_{\Omega}$ and $c_{\mathcal{D},q} = \delta(\mathcal{D})_{\{0, \dots, q-1\}}$.*

Proof. Let \mathcal{M} be a maximal Ω -matching (respectively $\{0, \dots, q-1\}$ -matching) in \mathcal{D} , then \mathcal{M} is perfect since $0 \in \Omega$ (respectively $0 \in \{0, \dots, q-1\}$). Thus $\text{card}(\mathcal{M}) \geq c_{\mathcal{D}}$ (respectively $\text{card}(\mathcal{M}) \geq c_{\mathcal{D},q}$) and therefore $\delta(\mathcal{D})_{\Omega} \geq c_{\mathcal{D}}$ (respectively $\delta(\mathcal{D})_{\{0, \dots, q-1\}} \geq c_{\mathcal{D},q}$).

Let $\mathcal{N} = \{T_1, \dots, T_s\} \subset \mathcal{D}$ (respectively $\mathcal{N} \subset \mathcal{D}$ and $\dim(T_i) \leq q-1$) such that $\bigcup_{i=1}^s T_i$ is spanning and $s = c_{\mathcal{D}}$ (respectively $s = c_{\mathcal{D},q}$). We claim that there is a maximal Ω -matching (respectively $\{0, \dots, q-1\}$ -matching) in \mathcal{D} with the same cardinality s . If $T_i \cap T_j = \emptyset$ for all indices $i, j \in \{1, \dots, s\}$, then \mathcal{N} is a maximal Ω -matching (respectively $\{0, \dots, q-1\}$ -matching) in \mathcal{D} . Suppose that there are indices $i, j \in \{1, \dots, s\}$, $i \neq j$, such that $T_i \cap T_j \neq \emptyset$. Set $T'_{j,i} = T_j - T_i$, then $T'_{j,i} \neq \emptyset$ since $T'_{j,i} = \emptyset$ implies that $T_j \subset T_i$ and also $\bigcup_{i=1, i \neq j}^s T_i$ is spanning. The last sentence contradicts the fact that $\text{card}(\mathcal{N}) = c_{\mathcal{D}}$ (respectively $\text{card}(\mathcal{N}) = c_{\mathcal{D},q}$). Note that $T'_{j,i} \subset T_j$, so it is a simplex with $\dim(T'_{j,i}) < \dim(T_j)$. Thus replacing consecutively in \mathcal{N} every T_j , with the property $T_i \cap T_j \neq \emptyset$ for some index $i \in \{1, \dots, s\}$, $i \neq j$, by $T'_{j,i}$, we arrive at a maximal Ω -matching (respectively $\{0, \dots, q-1\}$ -matching) \mathcal{M} in \mathcal{D} with $\text{card}(\mathcal{M}) = c_{\mathcal{D}}$ (respectively $\text{card}(\mathcal{M}) = c_{\mathcal{D},q}$). Consequently $\delta(\mathcal{D})_{\Omega} \leq c_{\mathcal{D}}$ (respectively $\delta(\mathcal{D})_{\{0, \dots, q-1\}} \leq c_{\mathcal{D},q}$). \square

Remark 3.4. From Theorem 2.8 we take that $\delta(\mathcal{D})_{\{0,1\}} = \delta(\mathcal{D})_{\{0\}} - \delta(\mathcal{D})_{\{1\}}$, so $c_{\mathcal{D},2} = \delta(\mathcal{D})_{\{0\}} - \delta(\mathcal{D})_{\{1\}}$.

The following theorems will make the connection between matchings of simplicial complexes and the radical of a lattice ideal.

Theorem 3.5. (Cf. [9].) For a lattice ideal $I_{L,\rho}$ with associated cone $\sigma = \text{pos}_{\mathbb{Q}}(A)$ we have $\delta(\mathcal{D}_{\sigma})_{\{0, \dots, q-1\}} \leq \text{ara}_{A,q}(I_{L,\rho})$ and $\delta(\mathcal{D}_{\sigma})_{\Omega} \leq \text{ara}_A(I_{L,\rho})$. In particular, $\mu(I_{L,\rho}) \geq \text{bar}(I_{L,\rho}) \geq \delta(\mathcal{D}_{\sigma})_{\{0\}} - \delta(\mathcal{D}_{\sigma})_{\{1\}}$ and $\text{ara}_A(I_{L,\rho}) \geq \gamma(\overline{\mathbb{G}(\mathcal{D}_{\sigma})})$.

Proof. The fact that $c_{\mathcal{D}_{\sigma}} \leq \text{ara}_A(I_{L,\rho})$ is already proved in Theorem 5.6 of [9]. Also Proposition 3.3 assures that $\delta(\mathcal{D}_{\sigma})_{\Omega} \leq \text{ara}_A(I_{L,\rho})$. Suppose, now, that $\text{ara}_{A,q}(I_{L,\rho}) = s$. This means that there exist A -homogeneous polynomials F_1, \dots, F_s such that $\text{rad}(I_{L,\rho}) = \text{rad}(F_1, \dots, F_s)$, where every F_i has at most q nonzero terms. Now Corollary 5.3 in [9] implies that $\bigcup_{i=1}^s \mathcal{D}_{\sigma}(F_i)$ is a spanning subcomplex of \mathcal{D}_{σ} and every $\mathcal{D}_{\sigma}(F_i)$ is a simplex with at most q vertices. Thus $s \geq c_{\mathcal{D}_{\sigma},q} = \delta(\mathcal{D}_{\sigma})_{\{0, \dots, q-1\}}$. Specifically $\mu(I_{L,\rho}) \geq \text{bar}(I_{L,\rho}) = \text{ara}_{A,2} \geq \delta(\mathcal{D})_{\{0,1\}} = \delta(\mathcal{D}_{\sigma})_{\{0\}} - \delta(\mathcal{D}_{\sigma})_{\{1\}}$ and $\text{ara}_A(I_{L,\rho}) \geq \delta(\mathcal{D}_{\sigma})_{\Omega} \geq \gamma(\overline{\mathbb{G}(\mathcal{D}_{\sigma})})$. \square

Theorem 3.6. [9] Let $I_{L,\rho}$ be a lattice ideal with associated cone σ . If F_1, \dots, F_s generate $\text{rad}(I_{L,\rho})$ up to radical then

- (i) the total number of monomials in the nonzero terms of the polynomials F_1, \dots, F_s is greater than or equal to the number of vertices $\delta(\mathcal{D}_{\sigma})_{\{0\}}$ of \mathcal{D}_{σ} ,
- (ii) the total number of A -homogeneous components in F_1, \dots, F_s is greater than or equal to the chromatic number of $\overline{\mathbb{G}(\mathcal{D}_{\sigma})}$.

4. Circuits of a vector configuration and applications

Let $A = \{\mathbf{a}_i \mid 1 \leq i \leq m\}$ be a vector configuration such that the cone $\sigma = \text{pos}_{\mathbb{Q}}(A)$ is strongly convex. Let \mathbf{D} be the matrix with columns the transposes of all the vectors in A and set $\mathcal{L}_A \subset \mathbb{Z}^m$ the lattice $\ker_{\mathbb{Z}}(\mathbf{D})$. Given two vectors \mathbf{u}, \mathbf{v} in \mathbb{Z}^m , we say that \mathbf{u} is *conformal* to \mathbf{v} if $\text{supp}(\mathbf{u}_+) \subset$

$\text{supp}(\mathbf{v}_+)$ and $\text{supp}(\mathbf{u}_-) \subset \text{supp}(\mathbf{v}_-)$. A nonzero vector $\mathbf{u} = (u_1, \dots, u_m) \in \mathcal{L}_A$ is called a *circuit* if its support $\text{supp}(\mathbf{u}) = \{i \mid u_i \neq 0\}$ is minimal with respect to inclusion and all the coordinates of \mathbf{u} are relatively prime, see [11]. The binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in K[x_1, \dots, x_m]$ is called also circuit. The notion of circuit comes from the oriented matroid theory. We denote by \mathcal{C}_A the set of circuits of the vector configuration A and put

$$\mathcal{C} := \{E \subset \{1, \dots, t\} \mid \text{supp}(\mathbf{u}_+) = E \text{ or } \text{supp}(\mathbf{u}_-) = E \text{ where } \mathbf{u} \in \mathcal{C}_A\}.$$

Let \mathcal{C}_{\min} be the set of minimal elements of \mathcal{C} .

Definition 4.1. We associate to a vector configuration A the simplicial complex Δ_A with vertices the elements of $\mathcal{C}_{\min} = \{E_1, \dots, E_k\}$. Let $T \subset \mathcal{C}_{\min}$ then $T \in \Delta_A$ if

$$\bigcap_{E_i \in T} \text{relint}_{\mathbb{Q}}(\sigma_{E_i}) \neq \emptyset.$$

Theorem 4.2. For a vector configuration A and $E_i, E_j \in \mathcal{C}_{\min}$ with $i \neq j$ we have that $\{E_i, E_j\} \in \Delta_A$ if and only if

- (i) $E_i \cap E_j = \emptyset$,
- (ii) there exist a circuit $\mathbf{u} \in \mathcal{C}_A$ such that $\text{supp}(\mathbf{u}_+) = E_i$, $\text{supp}(\mathbf{u}_-) = E_j$, and
- (iii) the dimension of the cone $\sigma_{E_i} \cap \sigma_{E_j}$ is equal to one.

Proof. (\Rightarrow) (i) The assumption $\{E_i, E_j\} \in \Delta_A$ implies that

$$\text{relint}_{\mathbb{Q}}(\sigma_{E_i}) \cap \text{relint}_{\mathbb{Q}}(\sigma_{E_j}) \neq \emptyset$$

and therefore there exist integral vectors $\mathbf{v}_1, \mathbf{v}_2$, both of them having nonnegative coordinates, such that $\text{supp}(\mathbf{v}_1) = E_i$, $\text{supp}(\mathbf{v}_2) = E_j$ and $\mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{L}_A$. If $E_i \cap E_j \neq \emptyset$, then either $\text{supp}((\mathbf{v}_1 - \mathbf{v}_2)_+) \subsetneq \text{supp}(\mathbf{v}_1) = E_i$ or $\text{supp}((\mathbf{v}_1 - \mathbf{v}_2)_-) \subsetneq \text{supp}(\mathbf{v}_2) = E_j$. Since $\mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{L}_A$, we can find, from Lemma 4.10 in [11], a circuit $\mathbf{u} \in \mathcal{C}_A$ which is conformal to $\mathbf{v}_1 - \mathbf{v}_2$. So $\text{supp}(\mathbf{u}_+) \subsetneq E_i$ or $\text{supp}(\mathbf{u}_-) \subsetneq E_j$, a contradiction to the minimality of E_i, E_j .

(ii) From (i) and its proof we have that there exists a vector $\mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{L}_A$ such that $\text{supp}((\mathbf{v}_1 - \mathbf{v}_2)_+) = E_i$ and $\text{supp}((\mathbf{v}_1 - \mathbf{v}_2)_-) = E_j$. Again there is a circuit $\mathbf{u} \in \mathcal{C}_A$ conformal to $\mathbf{v}_1 - \mathbf{v}_2$. From the minimality of the elements in \mathcal{C}_{\min} it follows that $\text{supp}(\mathbf{u}_+) = E_i$ and $\text{supp}(\mathbf{u}_-) = E_j$.

(iii) From the assumption $\{E_i, E_j\} \in \Delta_A$ we have that the $\sigma_{E_i} \cap \sigma_{E_j} \neq \{\mathbf{0}\}$, so it is at least one dimensional. Assume that the dimension is greater than one. Then there exist at least two vectors $\mathbf{v} = (v_1, \dots, v_m)$, $\mathbf{w} = (w_1, \dots, w_m)$ in \mathcal{L}_A such that $\text{supp}(\mathbf{v}) = \text{supp}(\mathbf{w}) = E_i \cup E_j$ and $\mathbf{a} = \deg_A(\mathbf{x}^{\mathbf{v}_+}) = \deg_A(\mathbf{x}^{\mathbf{v}_-})$, $\mathbf{b} = \deg_A(\mathbf{x}^{\mathbf{w}_+}) = \deg_A(\mathbf{x}^{\mathbf{w}_-})$ are linearly independent. Consider an index $t \in E_i$ and the nonzero vector $w_t \mathbf{v} - v_t \mathbf{w} \in \mathcal{L}_A$. Then t is not in $\text{supp}(w_t \mathbf{v} - v_t \mathbf{w})$, so there exist a circuit \mathbf{c} conformal to $w_t \mathbf{v} - v_t \mathbf{w}$. Thus $\text{supp}(\mathbf{c}) \subsetneq E_i \cup E_j$, since $\text{supp}(w_t \mathbf{v} - v_t \mathbf{w}) \subsetneq E_i \cup E_j$. But this fact contradicts the existence, from (ii) of this theorem, of a circuit \mathbf{u} such that $\text{supp}(\mathbf{u}) = E_i \cup E_j$.

(\Leftarrow) It follows from (ii). \square

Corollary 4.3. A set $T \subset \{E_1, \dots, E_k\}$ is an l -simplex, $l \geq 1$, of Δ_A if and only if the dimension of the cone $\bigcap_{E_i \in T} \sigma_{E_i}$ is equal to one.

Proof. If T is an l -simplex, then, from the definition of Δ_A , we have that $\bigcap_{E_i \in T} \sigma_{E_i} \neq \{\mathbf{0}\}$. Also for every $E_j \neq E_t \in T$ it holds

$$\bigcap_{E_i \in T} \sigma_{E_i} \subset \sigma_{E_j} \cap \sigma_{E_t}.$$

But, from Theorem 4.2, the dimension of $\sigma_{E_j} \cap \sigma_{E_t}$ equals one, so the cone $\bigcap_{E_i \in T} \sigma_{E_i}$ is one dimensional.

Conversely assuming that the dimension of $\bigcap_{E_i \in T} \sigma_{E_i}$ equals one we take that $\bigcap_{E_i \in T} \text{relint}_{\mathbb{Q}}(\sigma_{E_i}) \neq \emptyset$. Thus T belongs to Δ_A . \square

We call a vector configuration A *extremal* if the strongly convex rational polyhedral cone $\sigma = \text{pos}_{\mathbb{Q}}(A)$ is not generated by any proper subset of A , that means for $B \subsetneq A$ we have $\text{pos}_{\mathbb{Q}}(B) \subsetneq \text{pos}_{\mathbb{Q}}(A)$. Each vector in an extremal vector configuration A is an extreme vector of $\text{pos}_{\mathbb{Q}}(A)$ and there is only one extreme vector for each extreme ray of σ . The corresponding varieties $V(I_A)$ are called *extremal toric varieties*. Circuits of an extremal vector configuration reflect the geometry of the cone σ and actually provide all the information needed for constructing the simplicial complex \mathcal{D}_{σ} .

Lemma 4.4. For an extremal vector configuration $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and a set $E \subset \{1, \dots, m\}$ we have that the vector \mathbf{a}_j belongs to the cone σ_E if and only if $j \in E$.

Proof. Obviously $j \in E$ implies that $\mathbf{a}_j \in \sigma_E$. Conversely suppose that $\mathbf{a}_j = \sum_{i \in E} \lambda_i \mathbf{a}_i \in \sigma_E$. Set $S = \{i \in E \mid \lambda_i > 0\}$. Then $\mathbf{a}_j = \sum_{i \in S} \lambda_i \mathbf{a}_i$ and therefore, multiplying both parts of the previous equality with a vector that defines the face $\sigma_{\{j\}}$ of σ , we take that every vector \mathbf{a}_i , $i \in S$, belongs to the cone $\sigma_{\{j\}}$. Thus $j \in S$, since we take only one extreme vector for each extreme ray of σ . \square

Remark 4.5. From the previous lemma we have that $\sigma_{E'} \subset \sigma_E$ if and only if $E' \subset E$.

Theorem 4.6. For an extremal vector configuration A the simplicial complexes Δ_A and \mathcal{D}_{σ} are identical.

Proof. From the definitions of the simplicial complexes Δ_A and \mathcal{D}_{σ} it is enough to show that they have the same vertices. Let $\{\mathbb{E}_i\} \in \mathcal{D}_{\sigma}$, for an $i \in \{1, \dots, f\}$. Then $\sigma_{\mathbb{E}_i}$ is a minimal nonface of σ . The toric ideal I_A is equal to the radical of the ideal generated by \mathcal{C}_A , see [7]. Therefore, from Theorem 5.1 in [9], there is a circuit $\mathbf{u} \in \mathcal{C}_A$ such that $\text{cone}(\mathbf{x}^{\mathbf{u}+}) = \sigma_{\mathbb{E}_i}$. For the definition and results about the cone of a monomial see [9]. Recall that the $\text{cone}(\mathbf{x}^{\mathbf{u}+}) = \sigma_{\text{supp}(\mathbf{u}_+)}$, since in this case all vectors are extreme vectors of σ . Thus $\sigma_{\text{supp}(\mathbf{u}_+)} = \sigma_{\mathbb{E}_i}$ and therefore, from Remark 4.5, we have that $\text{supp}(\mathbf{u}_+) = \mathbb{E}_i$. Consequently $\mathbb{E}_i \in \mathcal{C}$ and we claim that also $\mathbb{E}_i \in \mathcal{C}_{\min}$. Suppose not, then there exist a circuit $\mathbf{v} \in \mathcal{C}_A$ such that $\text{supp}(\mathbf{v}_+) \subsetneq \text{supp}(\mathbf{u}_+)$. So $\sigma_{\text{supp}(\mathbf{v}_+)} \subsetneq \sigma_{\mathbb{E}_i}$, which implies that $\sigma_{\text{supp}(\mathbf{v}_+)}$ is a face \mathcal{F} of σ and therefore, multiplying the equation $\deg_A(\mathbf{x}^{\mathbf{v}+}) = \deg_A(\mathbf{x}^{\mathbf{v}-})$ with a vector $\mathbf{c}_{\mathcal{F}}$ defining \mathcal{F} , we arrive at a contradiction, since all vectors are extreme.

Conversely consider $E_i \in \mathcal{C}_{\min}$, for an $i \in \{1, \dots, k\}$. Then there is a circuit $\mathbf{u} \in \mathcal{C}_A$ such that $\text{supp}(\mathbf{u}_+) = E_i$. The cone σ_{E_i} is not a face of σ , since if it was, multiplying the equation $\deg_A(\mathbf{x}^{\mathbf{u}+}) = \deg_A(\mathbf{x}^{\mathbf{u}-})$ with a vector defining the face we arrive at a contradiction. If σ_{E_i} is not a minimal nonface of σ , then there is a minimal nonface $\sigma_{E'_i}$ of σ such that $\sigma_{E'_i} \subsetneq \sigma_{E_i}$ and therefore

$E'_i \subsetneq E_i$. According to the first part of this proof, since $\sigma_{E'_i}$ is a minimal nonface of σ , there exist a circuit $\mathbf{u}' \in \mathcal{C}_A$ such that $\text{supp}(\mathbf{u}'_+) = E'_i$. A contradiction to the fact that $E_i \in \mathcal{C}_{\min}$. \square

Corollary 4.3 combined with Theorem 4.6 give us a method for computing the simplicial complex \mathcal{D}_σ . We start with a set of nonzero integer vectors A in \mathbb{Z}^n . In case that a lattice L is given, the Smith's normal form computation corresponds to every lattice L such a set A . Then compute from A an extremal vector configuration B by keeping one vector in each extreme ray of the $\text{pos}_{\mathbb{Q}}(A)$ and forgetting the rest. Produce the circuits of the vector configuration B , which can be computed easily, see [3] or [4]. Create \mathcal{C} , next find \mathcal{C}_{\min} and then identify those circuits \mathbf{u} that both $\text{supp}(\mathbf{u}_+)$, $\text{supp}(\mathbf{u}_-)$ are in \mathcal{C}_{\min} . By looking at the A -degrees of the corresponding monomials, $\deg_A(\mathbf{x}^{\mathbf{u}_+}) = \deg_A(\mathbf{x}^{\mathbf{u}_-})$, from Theorem 4.2 and Corollary 4.3 we get all the simplices of $\Delta_B = \mathcal{D}_\sigma$.

There are known algorithms computing the chromatic number and the matching number of a graph, see, for example, [1,5,10]. We can take advantage of these algorithms to compute certain invariants of \mathcal{D}_σ . According to Theorem 3.5 the invariants $\delta(\mathcal{D}_\sigma)_{\{0,1\}}$ and $\gamma(\overline{\mathbb{G}(\mathcal{D}_\sigma)})$ provide lower bounds for the minimum number of generators, the binomial arithmetical rank and the A -homogeneous arithmetical rank of a lattice ideal. Additionally they provide information about the complexity of the problem of computing the arithmetical rank, see Theorem 3.6.

The next example shows how these algorithms can be applied, how the computation of these invariants help us determine the exact values of the binomial arithmetical rank and the A -homogeneous arithmetical rank and even give us information about the arithmetical rank in certain cases.

Example 4.7. (A toric ideal I_A with $\text{ara}(I_A) \neq \text{ara}_A(I_A) \neq \text{bar}(I_A) \neq \mu(I_A)$.) Consider the matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

Let A be the set of columns of \mathbf{D} , then the cone $\sigma = \text{pos}_{\mathbb{Q}}(A)$ is the cone over the three-dimensional cube. Note that A is an extremal vector configuration. The circuits of A are

$$\begin{aligned} \mathcal{C}_A = \{ & x_5x_8 - x_6x_7, x_3x_8 - x_4x_7, x_1x_8 - x_2x_7, x_2x_8 - x_4x_6, x_1x_8 - x_3x_6, \\ & x_1x_8 - x_4x_5, x_2x_7 - x_3x_6, x_2x_7 - x_4x_5, x_3x_6 - x_4x_5, x_1x_7 - x_3x_5, \\ & x_1x_6 - x_2x_5, x_1x_4 - x_2x_3, x_1x_8^2 - x_4x_6x_7, x_2x_7^2 - x_3x_5x_8, x_3x_6^2 - x_2x_5x_8, \\ & x_4^2x_5 - x_2x_3x_8, x_1^2x_8 - x_2x_3x_5, x_4x_5^2 - x_1x_6x_7, x_3^2x_6 - x_1x_4x_7, x_2^2x_7 - x_1x_4x_6 \}. \end{aligned}$$

Thus the elements of \mathcal{C}_{\min} are: $E_1 = \{1, 4\}$, $E_2 = \{1, 6\}$, $E_3 = \{1, 7\}$, $E_4 = \{1, 8\}$, $E_5 = \{2, 3\}$, $E_6 = \{2, 5\}$, $E_7 = \{2, 7\}$, $E_8 = \{2, 8\}$, $E_9 = \{3, 5\}$, $E_{10} = \{3, 6\}$, $E_{11} = \{3, 8\}$, $E_{12} = \{4, 5\}$, $E_{13} = \{4, 6\}$, $E_{14} = \{4, 7\}$, $E_{15} = \{5, 8\}$, $E_{16} = \{6, 7\}$. So $\text{Vertices}(\mathbb{G}(\mathcal{D}_\sigma)) = \{E_1, \dots, E_{16}\}$. The cardinality $\delta(\mathcal{D}_\sigma)_{\{0\}}$ of $\text{Vertices}(\mathbb{G}(\mathcal{D}_\sigma))$ is equal to 16. From the circuits it follows also that

$$\begin{aligned} \text{Edges}(\mathbb{G}(\mathcal{D}_\sigma)) = \{ & \{E_1, E_5\}, \{E_2, E_6\}, \{E_3, E_9\}, \{E_4, E_7\}, \{E_4, E_{10}\}, \{E_4, E_{12}\}, \\ & \{E_7, E_{10}\}, \{E_7, E_{12}\}, \{E_8, E_{13}\}, \{E_{10}, E_{12}\}, \{E_{11}, E_{14}\}, \{E_{15}, E_{16}\} \}. \end{aligned}$$

The matching number of $\mathbb{G}(\mathcal{D}_\sigma)$ is 8. Thus $\delta(\mathcal{D}_\sigma)_{\{0,1\}} = 16 - 8 = 8$, so $\text{bar}(I_A) \geq 8$. The chromatic number of $\overline{\mathbb{G}(\mathcal{D}_\sigma)}$ is equal to 7. Note also that \mathcal{D}_σ consists of only sixteen 0-simplices, twelve 1-simplices, four 2-simplices and one 3-simplex, namely $\{E_4, E_7, E_{10}, E_{12}\}$. Since

$$\deg_A(x_1x_8) = \deg_A(x_2x_7) = \deg_A(x_3x_6) = \deg_A(x_4x_5) = (2, 0, 0, 0).$$

It is easy to see that $\mathbb{D}(\mathbb{G}(\mathcal{D}_\sigma)) = \mathcal{D}_\sigma$, so, from Proposition 2.10, we have $\delta(\mathcal{D}_\sigma)_\Omega = \gamma(\overline{\mathbb{G}(\mathcal{D}_\sigma)}) = 7$. Therefore $\text{ara}_A(I_A) \geq 7$. While for the arithmetical rank of I_A we have the inequality $4 = \text{ht}(I_A) \leq \text{ara}(I_A) \leq \text{ara}_A(I_A)$.

Now, for the ideal I_A we have $\mu(I_A) = 9$. Also, from Theorem 3.6, we know that in the polynomials that generate I_A up to radical there must be at least 16 monomials with support specified by the Vertices($\mathbb{G}(\mathcal{D}_\sigma)$) in at least 7 A -homogeneous components. Using this information and the computer algebra system CoCoA (see [2]) we deduce that

- (i) $\text{bar}(I_A) = 8$, since $I_A = \text{rad}(x_5x_8 - x_6x_7, x_4x_7 - x_3x_8, x_2x_7 - x_3x_6, x_2x_8 - x_4x_6, x_1x_8 - x_4x_5, x_1x_7 - x_3x_5, x_1x_6 - x_2x_5, x_1x_4 - x_2x_3)$,
- (ii) $\text{ara}_A(I_A) = 7$, since $I_A = \text{rad}(x_5x_8 - x_6x_7, x_4x_7 - x_3x_8, x_1x_8 - x_4x_5 + x_2x_7 - x_3x_6, x_2x_8 - x_4x_6, x_1x_7 - x_3x_5, x_1x_6 - x_2x_5, x_1x_4 - x_2x_3)$, and
- (iii) $4 \leq \text{ara}(I_A) \leq 5$, since $I_A = \text{rad}((x_5x_8 - x_6x_7) + (x_4x_7 - x_3x_8), x_1x_8 - x_4x_5 + x_2x_7 - x_3x_6, (x_1x_4 - x_2x_3) + (x_2x_5 - x_1x_6), x_2x_8 - x_4x_6, x_1x_7 - x_3x_5)$.

Actually in the 5 polynomials above we have exactly 16 monomials in 7 A -homogeneous components. Note that these results are independent of the characteristic of the field K . In this example, as also for an infinite class of toric varieties studied in [9], the bounds given in Section 3 are sharp.

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References

- [1] N. Christofides, An algorithm for the chromatic number of a graph, *Comput. J.* 14 (1971) 38–39.
- [2] CoCoATeam, CoCoA: A system for doing computations in commutative algebra, available at <http://cocoa.dima.unige.it>.
- [3] J.A. De Loera, Triangulations of polytopes and computational algebra, PhD thesis, Cornell University, 1995.
- [4] P. Diaconis, D. Eisenbud, B. Sturmfels, Lattice walks and primary decomposition, in: B.E. Sagan, R.P. Stanley (Eds.), *Mathematical Essays in Honor of Gian-Carlo Rota*, Birkhäuser, Boston, 1998, pp. 173–193.
- [5] J. Edmonds, Paths, trees and flowers, *Canad. J. Math.* 17 (1965) 449–467.
- [6] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1994.
- [7] D. Eisenbud, B. Sturmfels, Binomial ideals, *Duke Math. J.* 84 (1996) 1–45.
- [8] T. Gallai, Über extreme Punkt- und Kantenmengen, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 2 (1959) 133–138.
- [9] A. Katsabekis, M. Morales, A. Thoma, Stanley–Reisner rings and the radicals of lattice ideals, *J. Pure Appl. Algebra* 204 (2006) 584–601.
- [10] S. Micali, V.V. Vazirani, An $o(\sqrt{|V|}|E|)$ algorithm for finding maximum matching in general graphs, in: *Proc. 21st Ann. Symp. on Foundations of Computer Science*, Syracuse, New York, 13–15 October, 1980, IEEE Comput. Soc. Press, Silver Spring, MD, 1980, pp. 17–27.
- [11] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Ser., vol. 8, Amer. Math. Soc., Providence, RI, 1995.